Algebraic methods for polynomial differential equations

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Talk based on the papers:

- B. Coll, A. Gasull, R. Prohens. Probability of existence of limit cycles for a family of planar systems. J. Differ. Equations, 373, 152–175. 2023.
- A. Gasull, V. Mañosa. Periodic orbits of discrete and continuous dynamical systems via Poincaré–Miranda theorem. Discrete Contin. Dyn. Syst. Ser. B, 25(2), 651-670. 2020.
- A. Gasull. Some open problems in low dimensional dynamical systems. SeMA J., 78, 233-269. 2021.
- A. Gasull and H. Giacomini. Effectiveness of the Bendixon-Dulac theorem, J. Differ. Equations 305, 347–367. 2021
- A. Gasull and H. Giacomini. Higgins–Selkov system revisited, In preparation 2024.

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Outline of the talk

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Goal of the talk

We will introduce the following methods:

- Resultant of two polynomials.
- Poincaré–Miranda Theorem.
- Bendixson–Dulac Theorem.

We will also apply them to study several problems for planar polynomial differential equations:

- Number of critical points.
- Non existence of periodic orbits.
- Number of limit cycles.

Outline of the talk

Resultants

- The resultant: definition.
- The discriminant.

Examples of application of the resultant and the discriminant:

- Casas-Alvero Conjecture.
- Maximum number of critical points.
- Sign of 1-parameter families of polynomials.

The resultant: definition

Given two polynomials $a = a(x), b = b(x) \in \mathbb{C}[x]$, with respective degrees ∂a and ∂b we are interested to have a simple criteria to know whether they have or not a common root (real or complex).

It is known that this property is controlled by a single computable (with a suitable determinant) complex number called the resultant of a and b and denoted by $\mathrm{Res}_x(a, b) = \mathrm{Res}(a, b; x)$.

We write $n = \partial a$, $m = \partial b$, and

$$a = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0, \quad b = b_m x^m + \cdots + b_2 x^2 + b_1 x + b_0.$$

We show it in next slide.

If $n = \partial a$ and $m = \partial b$, then $\mathrm{Res}_x(a, b)$ is the $(n+m) \times (n+m)$ determinant:

$$\mathrm{Res}_x(a,b) = \begin{vmatrix}
a_n & 0 & \dots & 0 & b_m & 0 & \dots & 0 \\
a_{n-1} & a_n & \dots & 0 & b_{m-1} & b_m & \dots & 0 \\
a_{n-2} & a_{n-1} & \dots & 0 & b_{m-2} & b_{m-1} & \ddots & 0 \\
\vdots & \vdots & \ddots & 0 & \vdots & \vdots & \ddots & 0 \\
\vdots & \vdots & \ddots & 0 & b_1 & \vdots & \ddots & 0 \\
\vdots & \vdots & \ddots & 0 & b_0 & b_1 & \vdots & b_m \\
\vdots & & \ddots & a_n & 0 & b_0 & \ddots & b_{m-1} \\
 & \vdots & & a_{n-1} & \vdots & 0 & \ddots & b_{m-2} \\
a_0 & a_1 & \dots & a_{n-2} & 0 & 0 & \ddots & \\
0 & a_0 & \ddots & \vdots & \vdots & 0 & \ddots & \vdots \\
0 & 0 & \ddots & a_1 & 0 & 0 & \ddots & b_1 \\
0 & 0 & \dots & a_0 & 0 & 0 & \dots & b_0
\end{vmatrix}$$

$$\longleftarrow \quad m \quad \longrightarrow \qquad \longleftarrow \quad n \quad \longrightarrow$$

The discriminant

One of the common uses of the resultant is to characterize when a polynomial $a(x)$ has a multiple root. If $n = \partial a$ it holds:

Proposition

Define the discriminant of $a \in \mathbb{C}[x]$ as

$$\triangle_x(a) = (-1)^{\frac{n(n-1)}{2}} \frac{1}{a_n} \operatorname{Res}_x(a, a').$$

Then a has some multiple root if and only if $\triangle_x(a) = 0$.

The proof is a simple consequence of the properties of the resultant and the fact that for a polynomial a, to have a multiple root is equivalent to say that a and a' share some root.

The factor $(-1)^{\frac{n(n-1)}{2}} \frac{1}{a_n}$ is simply for historical reasons. In particular, we want that for $a_2x^2 + a_1x + a_0$ it holds that $\Delta_x(a) = a_1^2 - 4a_2a_0$.

An application: Casas-Alvero Conjecture

Problem

The Casas-Alvero conjecture affirms that if a complex polynomial P of degree $n > 1$ shares roots with all its derivatives, $P^{(k)}, k = 1, 2 \ldots, n-1$, then there exist two complex numbers, a and $b \neq 0$, such that $P(z) = b(z-a)^n$.

Notice that, in principle, the common root between P and each $P^{(k)}$ might depend on k.

Casas-Alvero arrived to this problem at the turn of this century, when he was working trying to obtain an irreducibility criterion for two variable power series with complex coefficients.

Although several authors have got partial answers, to the best of our knowledge the conjecture remains open.

Casas-Alvero Conjecture-II

- For $n \leq 4$ we will prove it by using resultants.
- In 2006 it was proved by using Maple, that it is true for $n \leq 8$.
- Afterwards, it was proved that it holds when n is $p^m, 2p^m, 3p^m$ or $4p^m$, for some prime numbers p and $m \in \mathbb{N}$.
- The first cases left open are those where $n = 24, 28$ or 30.

A good reference:

- J. Draisma, J. P. de Jong. On the Casas-Alvero conjecture. Eur. Math. Soc. Newsl. 80 (2011), 29–33.

Casas-Alvero Conjecture: proof for $n = 4$

Write $P(x) = x^4 + bx^3 + cx^2 + dx + e$. The conditions of the conjecture can be algebraically written as:

$$C_1 = \mathrm{Res}_x(P, P') = 0, \quad C_2 = \mathrm{Res}_x(P, P'') = 0, \quad C_3 = \mathrm{Res}_x(P, P''') = 0.$$

It holds that

$$\begin{aligned} C_1 = &-27\, b^4 e^2 + 18\, b^3 cde - 4\, b^3 d^3 - 4\, b^2 c^3 e + b^2 c^2 d^2 + 144\, b^2 ce^2 \\ &- 6\, b^2 d^2 e - 80\, bc^2 de + 18\, bcd^3 + 16\, c^4 e - 4\, c^3 d^2 - 192\, bde^2 \\ &- 128\, c^2 e^2 + 144\, cd^2 e - 27\, d^4 + 256\, e^3, \end{aligned}$$

$$\begin{aligned} C_2 = &-1296\, b^4 e + 432\, b^3 cd - 96\, b^2 c^3 + 6912\, b^2 ce - 2016\, bc^2 d + 400\, c^4 \\ &- 10368\, bde - 5760\, c^2 e + 3456\, cd^2 + 20736\, e^2, \end{aligned}$$

$$C_3 = -3888\, b^4 + 20736\, b^2 c - 82944\, bd + 331776\, e.$$

From $C_3 = 0$ we get that $e = E := \frac{3}{256}\, b^4 - \frac{1}{16}\, b^2 c + \frac{1}{4}\, bd.$

Casas-Alvero Conjecture: proof for $n = 4$

We have to solve

$$D_1 := C_1\big|_{e=E} = 0 \quad D_2 := C_2\big|_{e=E} = 0.$$

With some more computations we get

$$\mathrm{Res}_c(D_1, D_2) = -\frac{113927664375}{17592186044416}\left(b^3 - 16\,d\right)^{16},$$
$$\mathrm{Res}_d(D_1, D_2) = \frac{22325625}{4294967296}\left(3\,b^2 - 8\,c\right)^{16}.$$

Hence

$$c = \frac{3}{8}b^2, \quad d = \frac{1}{16}b^3 \quad \text{and}$$
$$e = \frac{3}{256}\,b^4 - \frac{1}{16}\,b^2c + \frac{1}{4}\,bd = \frac{1}{256}b^4,$$

and, finally,

$$P(x) = \left(\frac{b}{4} + x\right)^4.$$

Outline of the talk

Critical points of a polynomial system: an example

The critical points of the polynomial ODE, $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$, coincide with number of real solutions of

$$\begin{cases} P(x,y) = 43x^6 + 61y^3 - 43y = 0, \\ Q(x,y) = 43y^6 + 61x^3 - 43x = 0. \end{cases}$$

From the properties of the resultant we know that each of the solutions is also a solutions of the following system, where notice that each of the new equations depends only on 1 variable.

$$\begin{cases} U(x) = \mathrm{Res}_y(P(x,y), Q(x,y)) = (43x^6 + 61x^3 - 43x)R_{30}(x) = 0, \\ U(y) = \mathrm{Res}_x(P(x,y), Q(x,y)) = (43y^6 + 61y^3 - 43y)R_{30}(y) = 0, \text{ where} \end{cases}$$

$$\begin{aligned} R_{30}(x) = {} & 11688200277601\,x^{30} - 16580935277527\,x^{27} + 11688200277601\,x^{25} + 23521791905329\,x^{24} - 33161870555054\,x^{22} \\ & + 66736246801166\,x^{21} + 11688200277601\,x^{20} - 94672350113282\,x^{18} - 49742805832581\,x^{17} + 66736246801166\,x^{16} \\ & + 145990836484815\,x^{15} + 70565375715987\,x^{14} - 189344700226564\,x^{13} + 28937431083381\,x^{12} - 33368123400583\,x^{11} \\ & + 11688200277601\,x^{10} - 111616150043574\,x^{9} + 111842107471016\,x^{7} + 124971066196115\,x^{6} + 11688200277601\,x^{5} \\ & - 270275883897806\,x^{4} + 94672350113282\,x^{3} + 95261172193489\,x^{2} - 66736246801166\,x + 11688200277601. \end{aligned}$$

Number of critical points of a planar system

By using for instance Sturm sequences it can be proved that $U(x)$ has exactly 6 real solutions:

$$x_0 = 0 < x_1 \approx 0.597 < x_2 \approx 0.689 < x_3 \approx 0.7403 < x_4 \approx 0.780 < x_5 \approx 0.816.$$

As a consequence we obtain that the initial system has at most 6×6 real solution and that the candidates to be a solution are $(x, y) = (x_i, x_j)$ varying i and j between 0 and 5.

Moreover, for each $j = 1, \dots, 5$ and any $\varepsilon > 0$, it is possible to find $\overline{x}_j, \underline{x}_j$ such that

$$\underline{x}_j < x_j < \overline{x}_j, \quad \text{with} \quad \overline{x}_j, \underline{x}_j \in \mathbb{Q}, \quad \text{and} \quad |\overline{x}_j - \underline{x}_j| < \varepsilon.$$

For instance

$$\underline{x}_1 = \frac{59}{100} < x_1 < \frac{3}{5} = \overline{x}_1, \quad \text{and} \quad \underline{x}_3 = \frac{74}{100} < x_3 < \frac{3}{4} = \overline{x}_3.$$

Number of critical points of a planar system

Some of the couples (x_i, x_j) are actual solutions of our initial system and some others are not.

Let us prove that (x_1, x_3) is not a solution. Recall that

$$\underline{x}_1 = \frac{59}{100} < x_1 < \frac{3}{5} = \overline{x}_1, \quad \text{and} \quad \underline{x}_3 = \frac{74}{100} < x_3 < \frac{3}{4} = \overline{x}_3.$$

We will prove that $P(x_1, x_3) = 43x_1^6 + 61x_3^3 - 43x_3 \neq 0$.

From the above inequalities get

$$\begin{aligned} P(x_1, x_3) &= 43x_1^6 + 61x_3^3 - 43x_3 < 43\overline{x}_1^6 + 61\overline{x}_3^3 - 43\underline{x}_3 \\ &= 43\left(\frac{3}{5}\right)^6 + 61\left(\frac{3}{4}\right)^3 - 43\left(\frac{74}{100}\right) = -\frac{4079417}{1000000} < 0, \end{aligned}$$

as we wanted to see.

Hence we have discarded this possibility.

Number of critical points of a planar system

Doing similar computations we get that apart from $(x, y) = (0, 0)$ which is trivially a solution of the initial systems, there are only 5 more candidates to be real solutions of:

$$\begin{cases} 43x^6 + 61y^3 - 43y = 0, \\ 43y^6 + 61x^3 - 43x = 0. \end{cases}$$

They are:

$$(x_1, x_5), (x_2, x_4), (x_3, x_3), (x_4, x_2), (x_5, x_1).$$

We need a different tool to ensure that they are actual solutions.
This will be proved by using a new tool, the so called Poincaré-Miranda theorem.
We will recall it in next slides.

Outline of the talk

Poincaré–Miranda Theorem

The Poincaré–Miranda Theorem is an extension of the classical Intermediate Value Theorem (or Bolzano's Theorem) to higher dimensions.

It was stated by H. Poincaré in 1883 and 1884, and proved by himself in 1886. In 1940, C. Miranda re-obtained the result as an equivalent formulation of Brouwer fixed point theorem:

- C. Miranda, Un'osservazione su un teorema di Brouwer, Boll. Un. Mat. Ital. (2) 3, (1940). 5–7.

In

- A. Gasull, V. Mañosa. Periodic orbits of discrete and continuous dynamical systems via Poincaré–Miranda theorem. Discrete Contin. Dyn. Syst. Ser. B, 25(2), 651-670. 2020.

we apply it to different problems of dynamical systems (discrete or continuous) to prove the existence of some periodic orbits.

Poincaré–Miranda Theorem

Theorem (Poincaré–Miranda)

Set $\mathcal{B} = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : L_i < x_i < U_i, 1 \le i \le n\}$. *Suppose that* $f = (f_1, f_2, \ldots, f_n) : \overline{\mathcal{B}} \to R^n$ *is continuous,* $f(\mathbf{x}) \neq \mathbf{0}$ *for all* $\mathbf{x} \in \partial\mathcal{B}$, *and for* $1 \le i \le n$,

$$f_i(x_1, \ldots, x_{i-1}, L_i, x_{i+1}, \ldots, x_n) \le 0 \ \text{ and}$$
$$f_i(x_1, \ldots, x_{i-1}, U_i, x_{i+1}, \ldots, x_n) \ge 0,$$

Then, there exists $\mathbf{s} \in \mathcal{B}$ *such that* $f(\mathbf{s}) = \mathbf{0}$.

Poincaré–Miranda Theorem in the plane

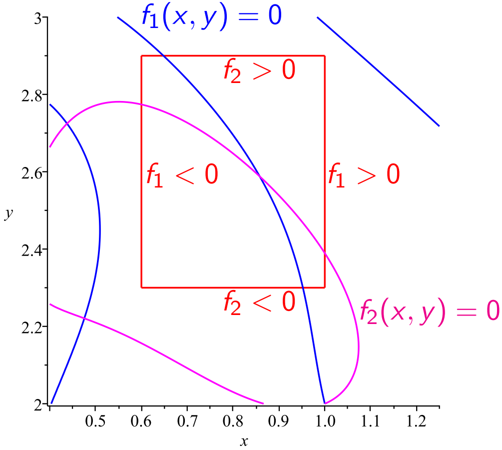


Figure: A Poincaré–Miranda box.

Outline of the talk

Kouchnirenko's conjecture

As we will see, the computation of the number of critical points of system

$$\begin{cases} \dot{x} = 43x^6 + 61y^3 - 43y, \\ \dot{y} = 43y^6 + 61x^3 - 43x, \end{cases}$$

that we started to study before is very related with the so called Kouchnirenko's conjecture.

Descartes' rule asserts that an 1-variable real polynomial with m monomials has at most $m-1$ simple positive real roots.

The Kouchnirenko's conjecture was posed to try to extend this rule to the several variables context. In the 2-variables case this conjecture said that:

> *A real polynomial system $f_1(x,y) = f_2(x,y) = 0$ would have at most $(m_1-1)(m_2-1)$ simple solutions with positive coordinates, where m_i is the number of monomials of each f_i.*

This conjecture was stated by A. Kouchnirenko in the late 70's, and published in the a A. G. Khovanskiĭ's paper in 1980.

Counterexamples to Kouchnirenko's conjecture

In 2000, B. Haas constructed a family of counterexamples given by two trimonomials, being their minimal degree 106.

In 2007 a much simpler family of counterexamples was presented by A. Dickenstein, J. M. Rojas, K. Rusek, J. Shih. The simplest one again formed by two trimonomials, but of degree 6.

Both have exactly 5 simple solutions with positive coordinates instead of the 4 predicted by the conjecture.

We give a similar counterexample by using PMT.

Proposition

The bivariate trinomial system

$$\begin{cases} P(x,y) := x^6 + \frac{61}{43}y^3 - y = 0, \\ Q(x,y) := y^6 + \frac{61}{43}x^3 - x = 0, \end{cases}$$

has 5 real simple solutions with positive entries.

Previous proved result

Notice that if P and Q are as in the previous proposition, we have already studied the solutions of the system

$$43P(x,y) = 0, \quad 43Q(x,y) = 0,$$

that is,

$$\begin{cases} 43x^6 + 61y^3 - 43y = 0, \\ 43y^6 + 61x^3 - 43x = 0. \end{cases}$$

Recall that we have already proved that apart from $(x,y) = (0,0)$ which is trivially a solution, there are only 5 more candidates to be real solutions:

$$(x_1, x_5), (x_2, x_4), (x_3, x_3), (x_4, x_2), (x_5, x_1),$$

for the previously given values.

Let us prove that they are indeed actual solutions by using Poincaré-Miranda theorem.

Proof of the proposition

It is not difficult to find numerically these 5 approximated solutions of the system. They are $(\widetilde{x}_1, \widetilde{x}_5)$, $(\widetilde{x}_2, \widetilde{x}_4)$, $(\widetilde{x}_3, \widetilde{x}_3)$, $(\widetilde{x}_4, \widetilde{x}_2)$, $(\widetilde{x}_5, \widetilde{x}_1)$, where $\widetilde{x}_1 = 0.59679166$, $\widetilde{x}_2 = 0.68913517$, $\widetilde{x}_3 = 0.74035310$, $\widetilde{x}_4 = 0.77980435$ and $\widetilde{x}_5 = 0.81602099$.

We consider the following 5 intervals, with $\widetilde{x}_i \in I_i$,

$$I_1 = \left[\frac{1}{2}, \frac{1619}{2500}\right], \; I_2 = \left[\frac{1619}{2500}, \frac{18}{25}\right], \; I_3 = \left[\frac{18}{25}, \frac{75857}{100000}\right],$$
$$I_4 = \left[\frac{75857}{100000}, \frac{4}{5}\right], \; I_5 = \left[\frac{4}{5}, \frac{83}{100}\right]$$

and prove that our system has 5 actual solutions (x_1, x_5), (x_2, x_4), (x_3, x_3), (x_4, x_2), (x_5, x_1), with $x_i \in I_i$.

Proof of the proposition

By Descartes' rule we know that there is exactly one simple positive real root of $P(x,x)$. The corresponding (x_3, x_3) is in in $I_3 \times I_3$.

By the symmetry of the system, if (x^*, y^*) is one of its solutions then (y^*, x^*) also is.

We will prove the existence of two more solutions (and so, their symmetric ones) by using PMT.

Proof of the proposition

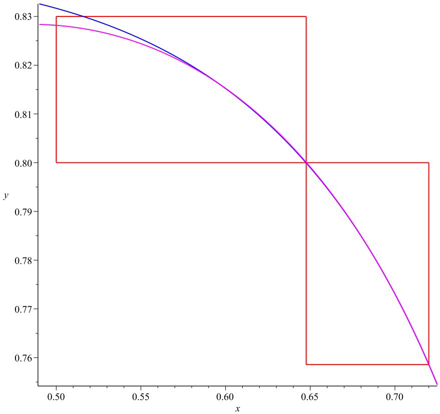


Figure: Intersection the curves defining the system. In red, the PM boxes $I_1 \times I_5$ and $I_2 \times I_4$

Proof of the proposition

We apply the PMT to the box $I_1 \times I_5 = \left[\frac{1}{2}, \frac{1619}{2500}\right] \times \left[\frac{4}{5}, \frac{83}{100}\right]$.
Consider the polynomials

$$P\left(\frac{1}{2}, y\right) = \frac{61}{43}\, y^3 - y + \frac{1}{2^6}$$

and

$$P\left(\frac{1619}{2500}, y\right) = \frac{61}{43}\, y^3 - y + \left(\frac{1619}{2500}\right)^6.$$

By computing their corresponding Sturm sequences we get that both have no roots in $[4/5, 83/100]$. Moreover $P(1/2, y) < 0$ and $P(1619/2500, y) > 0$ on this interval.

Proof of the proposition

Similarly we get that

$$Q\left(x, \frac{4}{5}\right) = \frac{61}{43}x^3 - x + \left(\frac{4}{5}\right)^6 < 0$$

and

$$Q\left(x, \frac{83}{100}\right) = \frac{61}{43}x^3 - x + \left(\frac{83}{100}\right)^6 > 0$$

on $[1/2, 1619/2500]$.

Hence, $I_1 \times I_5$ is under the hypotheses of the PMT, and our system has a solution (x_1, x_5) in this box.

Proof of the proposition

To prove that the 5 solutions are simple we compute

$$J(x,y) := \det \mathrm{D}(P,Q) = 36x^5y^5 - \frac{33489}{1849}x^2y^2 + \frac{183}{43}x^2 + \frac{183}{43}y^2 - 1.$$

Since $\mathrm{Res}(\mathrm{Res}(P,Q;x),\mathrm{Res}(P,J;x);y) \neq 0$, J does not vanish on the solutions (real or complex) of our system. Hence all their solutions are simple.

In fact, by joining the above result with the results proved in the previous lesson, we know that 5 is the exact number of solutions with positive entries and that these solutions together with $(0,0)$ are the only critical points of the polynomial ODE:

$$\begin{cases} \dot{x} = 43x^6 + 61y^3 - 43y, \\ \dot{y} = 43y^6 + 61x^3 - 43x. \end{cases}$$

Some open questions

In 2003, Li, Rojas and Wang proved that any bivariate trinomial system $m_1 = m_2 = 3$ has at most 5 real simple solutions with positive entries. These results have been tried to be extended for $m_1 = 3$ and m_2 arbitrary. In general:

Problem

Find a reasonable (or sharp) upper bound in terms of m_i for the maximum number of simple solutions with positive coordinates, for a real polynomial system $f_1(x, y) = f_2(x, y) = 0$, where m_i is the number of monomials of each f_i.

Not sharp upper bounds are known from the works of Khovanskiĭ.

Some open questions

Recall again that by using Descartes' rule it is easy to answer this last problem in one variable. Moreover, the $m-1$ corresponding to positive solutions goes to a global $2m-1$: $m-1$ positive roots, $m-1$ negative and, eventually, the root 0, that can be multiple, with any multiplicity.

It is natural to wonder if the following modified Kouchnirenko's bound works.

Problem

Is $(2m_1-1)(2m_2-1)$ the maximum number of simple solutions of a real polynomial system $f_1(x,y)=f_2(x,y)=0$, where m_i is the number of monomials of each f_i?

Of course there is a natural extension to several variables of the above bound.

Examples with $(2m_1 - 1)(2m_2 - 1)$ simple solutions

It is very easy to find examples of uncoupled systems having $(2m_1 - 1)(2m_2 - 1)$ simple solutions.

For instance, for $m_1 = m_2 = 3$, then $(2m_1 - 1)(2m_2 - 1) = 25$. Consider $(x^2 - 1)(x^2 - 4)x = x^5 - 5x^3 - x$. Then the system

$$\begin{cases} x^5 - 5x^3 - x = 0, \\ y^5 - 5y^3 - y = 0, \end{cases}$$

has the 25 simple solutions (x_i, x_j) with $x_i, x_j \in \{-2, -1, 0, 1, 2\}$. The system

$$\begin{cases} x^{5+r} - 5x^{3+r} - x^{1+r} = 0, \\ y^{5+s} - 5y^{3+s} - y^{1+s} = 0, \quad s > 0, r > 0, \end{cases}$$

has 16 simple solutions and 9 multiple ones.

An example with $(2m_1 - 1)(2m_2 - 1)$ solutions, but one non-simple

Another example with 25 solutions can be constructed from our counterexample:

$$\begin{cases} x^6 + \frac{61}{43}y^3 - y = 0, \\ y^6 + \frac{61}{43}x^3 - x = 0. \end{cases}$$

Taking $x \to x^2$ and $y \to y^2$ we consider:

$$\begin{cases} \left(x^{12} + \frac{61}{43}y^6 - y^2\right) y = x^{12}y + \frac{61}{43}y^7 - y^3 = 0, \\ \left(y^{12} + \frac{61}{43}x^6 - x^2\right) x = y^{12}x + \frac{61}{43}x^7 - x^3 = 0, \end{cases}$$

which has $4 \times 5 = 20$ solutions, 5 in each quadrant, plus 5 more on the axis: $(0,0), (\pm x^*, 0), (0, \pm y^*)$, for some x^* and y^*. Again 25 solutions, but here $(0,0)$ is not a simple solution.

Outline of the talk

Bendixson–Dulac theorem

Recall the version of the Bendixson—Dulac theorem for multiply connected regions:

Theorem

Consider a $\mathcal{C}^1$ *planar differential system*

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

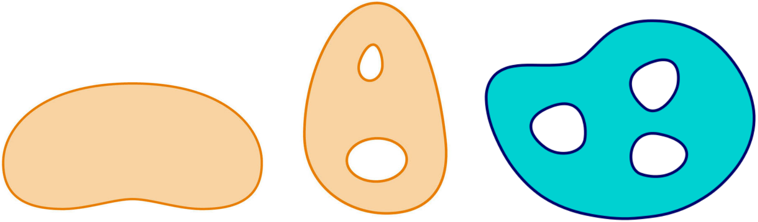
defined on $\mathcal{U} \subset \mathbb{R}^2$*, an open connected subset such that* $\mathbb{R}^2 \setminus \mathcal{U}$ *has* $\ell = \ell(\mathcal{U})$ *bounded components, and denote by* $X = (P, Q)$ *its associated vector field. Let* $B : \mathcal{U} \to \mathbb{R}$ *be a* $\mathcal{C}^1$ *function such that*

$$\operatorname{div}(BX) = (BP)_x + (BQ)_y$$

does not change sign and vanishes only on a null measure set. Then, the system has at most ℓ *limit cycles in* $\mathcal{U}$.

Bendixson–Dulac theorem in pictures

In a few words, from Bendixson-Dulac theorem we obtain the maximum number of limit cycles in $\mathcal{U}$ when the divergence does not change sign on $\mathcal{U}$, according the shape of the set $\mathcal{U}$.



No limit cycles   2 limit cycles   3 limit cycles

Bendixson–Dulac theorem. Definition of $L(V)$

Given an open connected subset $\mathcal{U} \subset \mathbb{R}^2$, with finitely many holes, we have denoted by $\ell = \ell(\mathcal{U})$ this number of holes, that is, the number of bounded components of $\mathbb{R}^2 \setminus \mathcal{U}$. Notice that if $\mathcal{U}$ is simply connected then $\ell(\mathcal{U}) = 0$.

Definition

Given a function $V : \mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^1$ we will say that it is admissible if:

(i) The vector ∇V vanishes on $\{V(x, y) = 0\}$ at finitely many points.

(ii) The set $\{V(x, y) = 0\}$ has finitely many connected components.

(iii) The set $\mathbb{R}^2 \setminus \{V(x, y) = 0\}$ has j connected components, $\mathcal{U}_i, i = 1, 2, \ldots, j$, and for all of them $\ell(\mathcal{U}_i) < \infty$.

Associated to V, we define the non negative integer number

$$L(V) := \sum_{i=1}^{j} \ell(\mathcal{U}_i).$$

Bendixson–Dulac theorem

Theorem (A version of Bendixson–Dulac theorem)

Consider a $\mathcal{C}^1$ *planar differential system*

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

and denote by $X = (P, Q)$ *its associated vector field. Let* $V : \mathbb{R}^2 \to \mathbb{R}$ *be an admissible function such that there exists* $s \in \mathbb{R}^+$ *for which the function*

$$M_s := M_{s,V} = \frac{\partial V}{\partial x} P + \frac{\partial V}{\partial y} Q - s \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) V$$

does not change sign and vanishes only on a null measure set, not invariant by the flow of X*. Then the system has at most* $L_X(V) := N + L(V)$ *limit cycles, where* N *is the number of periodic orbits of* X *contained in the set* $\mathcal{V} = \{V(x, y) = 0\}$ *and* $L(V)$ *the introduced computable number that depends on the shape of the set* $\{V(x, y) = 0\}$.

Bendixson–Dulac theorem, 2nd version. Idea of the proof

In a few words we apply the classical Bendixson–Dulac theorem with $B = |V|^{-1/s}$ to each of the connected components of $\mathbb{R}^2 \setminus \{V(x,y) = 0\}$. The key points are:

- The formula:

$$\operatorname{div}\left(|V|^{-1/s} X\right) = -\frac{1}{s}\operatorname{sign}(V)|V|^{-1/s-1} M_s$$

  where

$$M_s = M_{s,V} = \frac{\partial V}{\partial x} P + \frac{\partial V}{\partial y} Q - s\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V.$$

- The fact that

$$M_s\Big|_{V=0} = \frac{\partial V}{\partial x} P + \frac{\partial V}{\partial y} Q$$

  does not change sign implies that the periodic orbits of the system not contained in $\{V(x,y) = 0\}$ can not cut this set.

A key point in BD Theorem: control of the sign of a function

In next two situations the sign of a function gives dynamical properties of a phase portrait of a planar differential equation

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y).$$

- Lyapunov approach: $W : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that

  $\dot{W} = W_x P + W_y Q$ does not change sign implies no periodic orbits

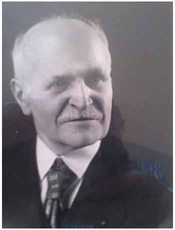
- Bendixson-Dulac approach: $B : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that

  $\mathrm{div}(BP, BQ) = \dot{B} + \mathrm{div}(P, Q)B$ does not change sign controls the p.o.

SIGN OF A FUNCTION $\Longrightarrow$ NUM. OF PERIODIC ORBITS

Aleksandr M. Lyapunov (1857-1919)

Ivar Bendixson (1861-1935)



Henri Dulac (1870-1955)

An example: fixed sign of 1-parameter families of polynomials

Lemma

Let

$$G_b(x) = g_n(b)x^n + g_{n-1}(b)x^{n-1} + \cdots + g_1(b)x + g_0(b),$$

be a family of real polynomials depending also polynomially on a real parameter b. Suppose that there exists an open interval $I \subset \mathbb{R}$ such that:

(i) *There is some $b_0 \in I$, such that $G_{b_0}(x) > 0$ on $\mathbb{R}$.*

(ii) *For all $b \in I$, $\triangle_x(G_b) \neq 0$.*

(iii) *For all $b \in I$, $g_n(b) \neq 0$.*

Then for all $b \in I$, $G_b(x) > 0$ on $\mathbb{R}$.

Proof of the lemma

$$G_b(x) = g_n(b)x^n + g_{n-1}(b)x^{n-1} + \cdots + g_1(b)x + g_0(b).$$

The key point of the proof is that the roots (real and complex) of G_b depend continuously of b, because $g_n(b) \neq 0$. Moreover:

- Hypothesis $g_n(b) \neq 0$ prevents that moving b some root appears from infinity.
- On the other hand if moving b some new real roots appear from $\mathbb{C}$, they do appear trough a multiple real root that is detected by the vanishing of $\triangle_x(G_b)$. Since by hypothesis (ii), $\triangle_x(G_b) \neq 0$ no real root appears in this way.

Hence, for all $b \in I$, the number of real roots of any G_b is the same. $\square$

Outline of the talk

Non existence of limit cycles

We study the non existence of periodic orbits of the celebrated Higgins–Selkov system. Recall that it is a model of glicolysis and it is

$$\begin{cases} \dot{x} = 1 - xy^2, \\ \dot{y} = ay(xy - 1), \end{cases}$$

where a is a real positive parameter. In

J. C. Artés, J. Llibre and C. Valls, Dynamics of the Higgins-Selkov and Selkov systems. Chaos Solitons Fractals 114, (2018) 145–150.

it is addressed the following conjecture: For $a \in \mathbb{R}$, the above system has limit cycles only when $a \in (1, a^*)$ and $a^* \in (1.23, 1.24)$. This conjecture is supported by several analytic and numerical results presented in that paper.

In particular, the authors prove that for $a \in (-\infty, 0] \cup [3, \infty)$ the system does not have limit cycles. We will extend the region of non existence of limit cycles.

Non existence of limit cycle for Higgins–Selkov system

We will prove next result, by using the Bendixson–Dulac approach.

Theorem

The Higgins–Selkov system

$$\begin{cases} \dot{x} = 1 - xy^2, \\ \dot{y} = ay(xy - 1), \end{cases}$$

does not have limit cycles for $a \in (\infty, 1] \cup [5/4, \infty)$

Recall that the conjectured region without limit cycles is $a \in (\infty, 1] \cup [a^*, \infty)$ with $a^* \in (1.23, 1.24)$.

With the same method and much more involved computations we can prove that the system does not have limit cycles when

$$a \in (\infty, 1] \cup [1.23907, \infty).$$

Proof of non existence of limit cycle for HS system

The periodic orbits of the HS system

$$\begin{cases} \dot{x} = 1 - xy^2, \\ \dot{y} = ay(xy - 1), \end{cases}$$

must be contained in the first quadrant $\mathcal{Q} = \{x > 0 \text{ and } y > 0\}$.
For convenience, in $\mathcal{Q}$, we take the new variables $u = xy$ and $v = y$, where for the sake of simplicity, we keep the old names for the variables,

$$\begin{cases} \dot{x} = y + ax(x-1) - xy^2 =: P(x,y), \\ \dot{y} = ay(x-1) =: Q(x,y), \end{cases}$$

Next, we will choose suitable different $V's$ in Bendixson–Dulac Theorem such that the corresponding M_1,

$$M_1 = \frac{\partial V}{\partial x} P + \frac{\partial V}{\partial y} Q - \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) V.$$

does not change sign on $\mathcal{Q}$.

Non existence of limit cycle for HS system. Case $a \leq 0$.

By taking

- $V(x,y) = y^3$, some computations give that $M_1(x,y) = y^3(y^2 - a)$.

Note that:

- The set $\mathcal{V} = \{V(x,y) = 0\} = \{y = 0\}$, does not contain periodic orbits of system,
- All the connected components of $\mathbb{R}^2 \setminus \mathcal{V}$ are simply connected,
- When $a \leq 0$ the function M_1 does not change sign on Ω.

Hence, by applying the Bendixson–Dulac Theorem:

For $a \leq 0$ the Higgins–Selkov system does not have periodic orbits.

Non existence of limit cycle for HS system. Case $0 \leq a \leq 1$.

By taking

- $V(x,y) = y^2\big(y + 2a(x-1)\big)$, some computations give that

$$M_1(x,y) = y^3\big((y-a)^2 + a(1-a)\big).$$

Note that:

- The set $\mathcal{V} = \{V(x,y) = 0\}$ does not contain periodic orbits of system,
- All the connected components of $\mathbb{R}^2 \setminus \mathcal{V}$ are simply connected,
- When $0 \leq a \leq 1$ the function M_1 does not change sign on Ω.

Hence, by applying the Bendixson–Dulac Theorem:

For $0 \leq a \leq 1$ the Higgins–Selkov system does not have periodic orbits.

Non existence of limit cycle for HS system. Case $a \geq 5/4$.

By taking

- $V(x,y) = y\big(v(x) - 4xy\big)$, with $v(x) = (k+5)\big(1-(k+1)x\big)$, and writing $a = (k+5)(k+1)^2/4$, then some computations give that

$$M_1(x,y) = \frac{k+1}{4}\left(\big(2y - v(x)\big)^2 + k(k+5)^2\right).$$

Note that:

- The set $\mathcal{V} = \{V(x,y) = 0\}$ does not contain periodic orbits of system,
- All the connected components of $\mathbb{R}^2 \setminus \mathcal{V}$ are simply connected,
- When $a \geq 5/4$ it is easy to see that there exists $k \geq 0$ such that $a = (k+5)(k+1)^2/4$. In particular, for $a = 5/4$, $k = 0$. Thus, when $a \geq 5/4$ the function M_1 does not change sign on Ω.

Hence, by applying the Bendixson–Dulac Theorem:

For $a \geq 5/4$ the Higgins–Selkov system does not have periodic orbits.

More about the case $a \geq 5/4$.

In this case it has been a key point to write

$$a = \frac{(k+5)(k+1)^2}{4}.$$

This type of tricks, that are performed to avoid square roots and cumbersome expressions, can be applied to several problems in dynamical systems and are explained and developed with more detail in:

- A. Gasull, T. Lázaro and J. Torregrosa, Rational parameterizations approach for solving equations in some dynamical systems problems. Qual. Theory Dyn. Syst. 18, (2019) 583–602.

Non existence of limit cycle for HS system. How to find V?

The main ideas follow from the methods developed in the papers

- A. Gasull and H. Giacomini. Some applications of the extended Bendixson–Dulac theorem. In Progress and challenges in dynamical systems, vol. 54 of Springer Proc. Math. Stat. Springer, Heidelberg, 2013, pp. 233–252.
- A. Gasull and H. Giacomini. Effectiveness of the Bendixon-Dulac theorem, J. Differ. Equations 305, (2021) 347–367.

In this case, it consists on considering $V(x,y) = y^m P_2(x,y)$, where the factor y appears because $y = 0$ is an invariant curve of the system, P_2 is second degree polynomials with free coefficients, and $s > 0$ and m are also free parameters. Then all these parameters are determined by imposing that M_s has some special shape that makes affordable to impose that it keeps sign on $\mathcal{Q}$.

Usually, the computation of several resultants or discriminants is one the more used tools to control the signs of the functions M_s.

Non existence of limit cycle for HS system. A final improvement

Until now we have proved non existence of periodic orbits for $a \geq 5/4 = 1.25$. To improve this result, showing that the same holds for

$$a \geq 1.23907$$

we really need much more computations.

We have achieved this result by searching for more complicated functions V. More concretely, we have considered functions of the form

$$V(x,y) = y^m P_2(x,y) \exp\left(\frac{Q_2(x,y)}{y}\right),$$

that also contain an exponential factor of the system.

Uniqueness of limit cycle for HS system

We are now working on proving the existence and uniqueness of the limit cycle for the Higgins–Steklov system when $b \in (0, b^*]$, by using also Bendixson–Dulac Theorem.
This result has been already proved in

- H. Chen and Y. Tang, Proof of Artés–Llibre-Valls's conjectures for the Higgins–Selkov and the Selkov systems. J. Differ. Equations 266, (2019) 7638–7657.

by transforming the system into a Liénard equation.

In next slides we present another system for which Bendixson–Dulac Theorem gives effective criteria for controlling the number of limit cycles.

Number of limit cycles

We give, under some testable conditions, an upper bound of the number of limit cycles for system system

$$\begin{cases} \dot{x} = af(x) + by, \\ \dot{y} = cf(x) + dy, \end{cases}$$

where a, b, c, d are real parameter, f is smooth and $f(0) = 0$.
In particular we prove

Consider system

$$\begin{cases} \dot{x} = ax^{2n-1} + by, \\ \dot{y} = cx^{2n-1} + dy, \end{cases}$$

where $n > 1$ is an integer. It has at most one limit cycle and, when it exists, it is hyperbolic. Moreover, it exists if and only if $ad - bc > 0$ and $ad < 0$ and its stability is given by the sign of $-d$.

Number of limit cycles

Theorem

Consider system

$$\dot{x} = af(x) + by, \quad \dot{y} = cf(x) + dy,$$

where f is smooth and $f(0) = 0$. Assume that

$$R(x) = 2af'(x)F(x) - a(f(x))^2 - dxf(x) + 2dF(x)$$

does not change sign and vanishes at isolated points, where $F' = f$ and $F(0) = 0$. Let K be the number of bounded intervals (counting also intervals degenerated to a point as intervals) of the closed set

$$\{x \in \mathbb{R} : \Delta(x) = (af(x) + dx)^2 - 8(ad - bc)F(x) \geq 0\}.$$

Then the system has at most K limit cycles, all of them hyperbolic.

Idea of the proof:

We apply Bendixson–Dulac Theorem with $s = 1$, that is

$$M(x, y) = M_1(x, y) = V_x P + V_y Q - \mathrm{div}(P, Q).$$

with the idea idea of searching for a function

$$V(x, y) = y^2 + v(x)y + w(x),$$

for some v and w, such that when we compute M with $(P, Q) = (af(x) + by, cf(x) + dy)$, we obtain that M depends only on x.
Some computations give that

$$\begin{aligned} M(x, y) = & \big(d + bv'(x) - af'(x)\big)y^2 \\ & + \big(af(x)v'(x) + bw'(x) + 2cf(x) - af'(x)v(x)\big)y \\ & af(x)w'(x) + cf(x)v(x) - af'(x)w(x) - dw(x). \end{aligned}$$

Idea of the proof:

$$
\begin{aligned}
M(x,y) =&\big(d + bv'(x) - af'(x)\big)y^2 \\
&+ \big(af(x)v'(x) + bw'(x) + 2cf(x) - af'(x)v(x)\big)y \\
&af(x)w'(x) + cf(x)v(x) - af'(x)w(x) - dw(x).
\end{aligned}
$$

Hence, to achieve our goal, we can choose v and w as any solution of the differential equations obtained by equating the coefficients of M of y and y^2 to zero. More concretely, we take

$$
v(x) = \frac{a}{b}f(x) - \frac{d}{b}x, \quad w(x) = \frac{2(ad-bc)}{b^2}F(x) - \frac{ad}{b^2}xf(x).
$$

Then

$$
M(x,y) = \frac{(bc-ad)}{b^2}R(x),
$$

where recall that, by hypothesis, R does not change sign and vanishes only at isolated points.

Idea of the proof:

In short we have found v and w such that by taking

$$V(x,y) = y^2 + v(x)y + w(x),$$

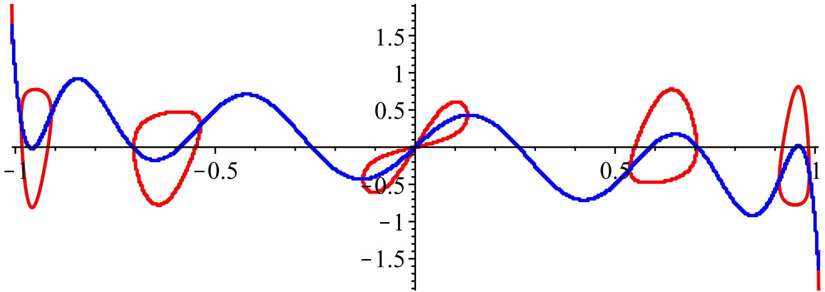
$$\operatorname{div}\left(\frac{P(x,y)}{V(x,y)}, \frac{Q(x,y)}{V(x,y)}\right) = \frac{R(x,y)}{V^2(x,y)} = \frac{(bc-ad)}{b^2}\frac{R(x)}{V^2(x,y)}.$$

When R does not change sign and vanishes at isolated points we can apply Bendixson-Dulac Theorem. We know that:

- The limit cycles do not cut the set $\{V(x,y)=0\}$,
- the maximum number of limit cycles is given by the number of holes of $\mathbb{R}^2 \setminus \{V(x,y)=0\}$.

Since V is quadratic on y, the shape of $V(x,y)=0$ can be easily studied. This number is controlled by the number of intervals of the discriminant of $V(x,y)$ with respect to y, that is the $\Delta(x)$ of the statement.

Idea of the proof:



In red, an example of set $\{V(x,y)=0\}$, for which $K=5$, corresponding to $(a,b,c,d)=(-1,-1,-1,1)$ and $f(x)=T_{11}(x)$, the 11-th Chebyshev polynomial.

The blue curve is simply the plot of the symmetry curve of $V(x,y)=0$ that also writes as

$$y=\frac{dx-af(x)\pm\sqrt{\Delta(x)}}{2b}.$$

Uniqueness of the limit cycle

Corollary

Consider system

$$\dot{x} = af(x) + by, \quad \dot{y} = cf(x) + dy,$$

and assume that

$$R(x) = 2af'(x)F(x) - a(f(x))^2 - dxf(x) + 2dF(x)$$

does not change sign and vanishes at isolated points. Assume also that the origin is the only equilibrium point of the system. Then it has at most one limit cycle, and when it exists, it is hyperbolic.

In this case, the uniqueness of the critical point implies that $\mathbb{R}^2 \setminus \{V(x, y) = 0\}$ has at most 1 hole.

Uniqueness of the limit cycle

Corollary

Consider system

$$\dot{x} = ax^{2n-1} + by, \quad \dot{y} = cx^{2n-1} + dy,$$

where $n > 1$ is an integer. It has at most one limit cycle and, when it exists, it is hyperbolic. Moreover, it exists if and only if $ad - bc > 0$ and $ad < 0$ and its stability is given by the sign of $-d$.

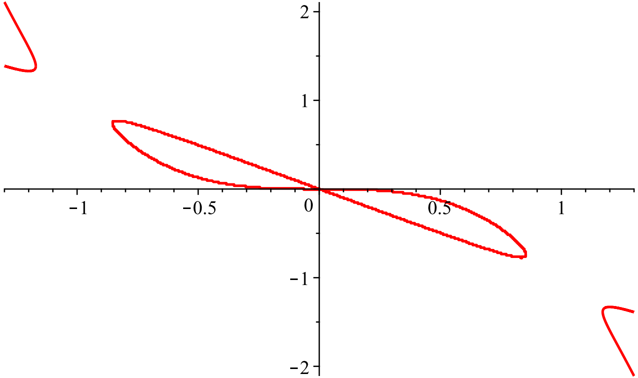
Idea of the proof:

For this system:

- When $ad > 0$, it does not have periodic orbits. This follows from the classical Dulac criterion because the divergence of the vector field is $(2n-1)ax^{2(n-1)} + d$ and does not change sign.
- Case $ad < 0$. Then
$$R(x) = \frac{(n-1)}{n} x^{2n}(ax^{2(n-1)} - d)$$
and hence this function does not change sign. Finally we need to compute the number K of bounded intervals where $\Delta(x) \geq 0$. In this case
$$\Delta(x) = x^2\Big(\frac{a^2}{b^2}x^{8(n-1)} + \frac{(4n-6)ad + 4bc}{(2n-1)b^2}x^{4(n-1)} + \frac{d^2}{b^2}\Big),$$
and it is not difficult to prove that $K = 1$ and the uniqueness of the limit cycle follows

Idea of the proof:

The shape of $\{V(x, y) = 0\}$ is illustrated in next figure when $(a, b, c, d) = (1, -1, -1.05, -1)$. In all cases the proof of uniqueness of the limit cycle is similar.



Research Centers cups collection in my office

Thank you very much for your attention