Higher order reduction methods for systems with multiple timescales

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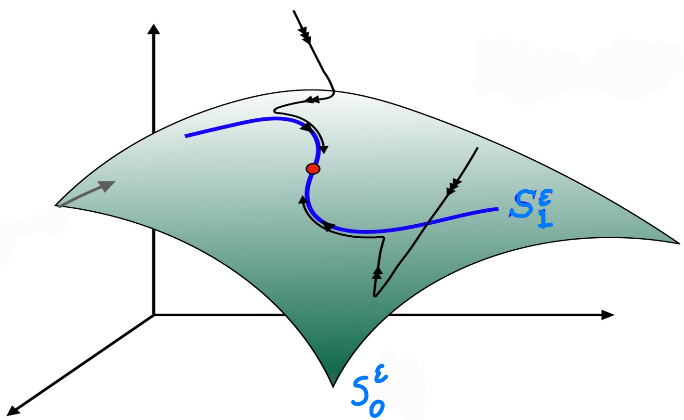
Multiple timescale systems

$$\dot{x} = F_0(x) + \varepsilon F_1(x) + \varepsilon^2 \ldots$$



Will see examples in:

* Reaction networks
* Coupled oscillators

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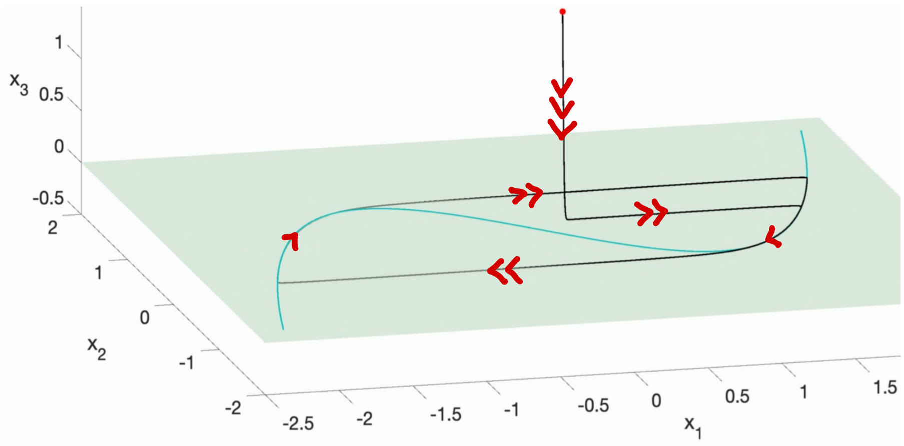
Example: embedded relaxation oscillation

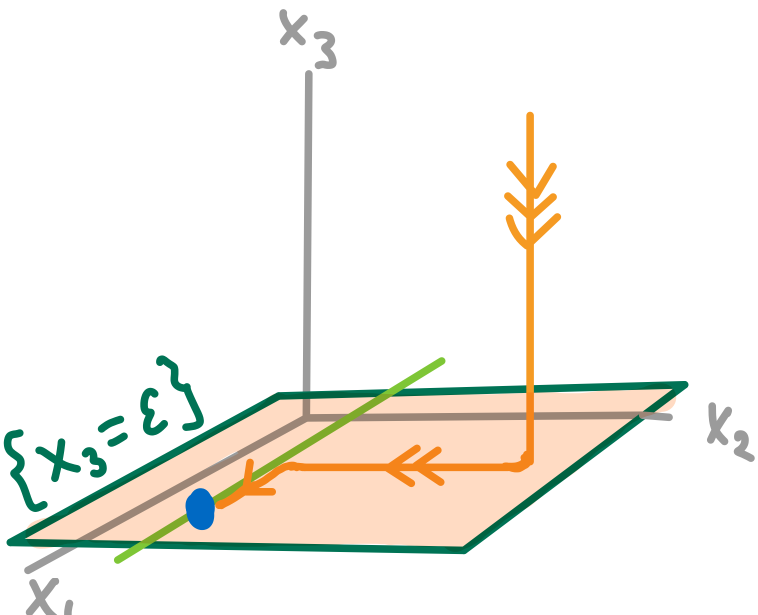
$$\begin{cases} \dot{x}_1 = \varepsilon(x_2 + x_1 - x_1^3/3) \\ \dot{x}_2 = -\varepsilon x_3 \\ \dot{x}_3 = -x_3 + \varepsilon x_1 \end{cases}$$



Another system with a hidden timescale

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -x_3 \end{pmatrix} + \varepsilon \begin{pmatrix} a_1 x_3 - a_2 x_1 x_2 \\ a_3 x_3 - a_4 x_2 \\ 1 \end{pmatrix}$$

$$\begin{tikzcd} & X_2 \arrow[dr, dash, "\varepsilon"] \arrow[dr, phantom, "\dashv"] & \\ X_3 \arrow[ur, "\varepsilon"] \arrow[rr, "\varepsilon"] & & X_1 \end{tikzcd}$$



"Nonstandard" slow dynamics:

$$\frac{1}{\varepsilon}\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -a_2 x_1 x_2 \\ -a_4 x_2 \end{pmatrix} + \varepsilon \begin{pmatrix} a_1 \\ a_3 \end{pmatrix}$$

Infraslow dynamics:

$$\frac{1}{\varepsilon^2}\frac{dx_1}{dt} = a_1 - \frac{a_2 a_3}{a_4} x_1$$

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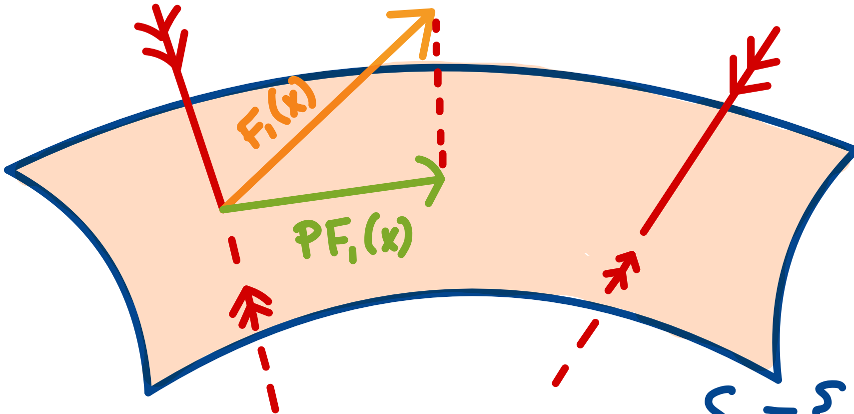
Reminder: Fénichel-Tikhonov's theorem



$$\dot{x} = F_0(x) + \varepsilon F_1(x) + \varepsilon^2 \ldots$$

$$S_0 = \{ F_0(x) = 0 \}$$

THM: S_0 persists as S_ε if it is normally hyperbolic; the "reduced" slow dynamics is given by

$$\frac{1}{\varepsilon} \dot{x} = PF_1(x) + \varepsilon \ldots \quad (x \in S_0).$$

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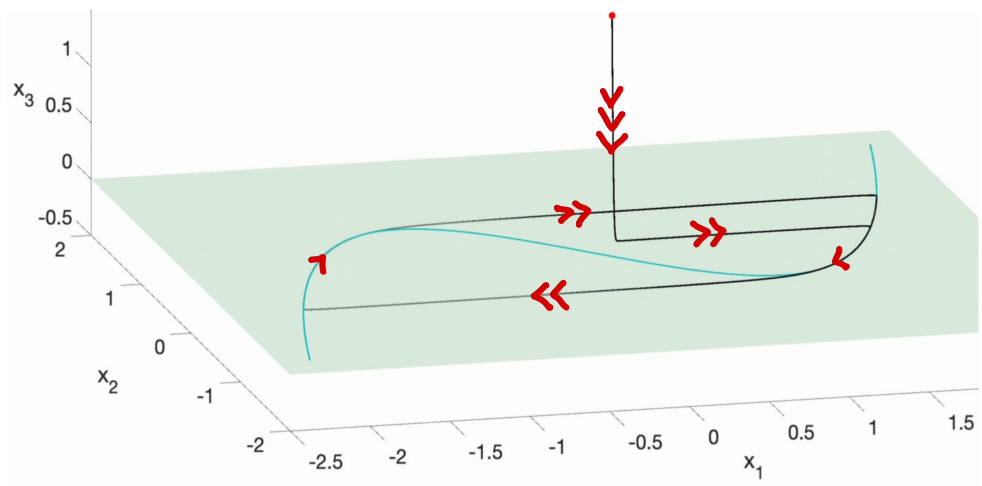
$$\begin{cases} \dot{x}_1 = \varepsilon\left(x_2 + x_1 - x_1^3/3\right) \\ \dot{x}_2 = -\varepsilon x_3 \\ \dot{x}_3 = -x_3 + \varepsilon x_1 \end{cases}$$

Tikhonov

$$\begin{cases} \dot{x}_1 = \varepsilon\left(x_2 + x_1 - x_1^3/3\right) \\ \dot{x}_2 = 0 \end{cases}$$

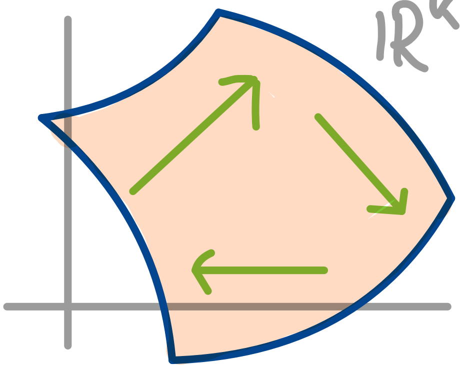
Actually

$$\begin{cases} \dot{x}_1 = \varepsilon\left(x_2 + x_1 - x_1^3/3\right) \\ \dot{x}_2 = -\varepsilon^2 x_1 + \mathcal{O}(\varepsilon^3) \end{cases}$$

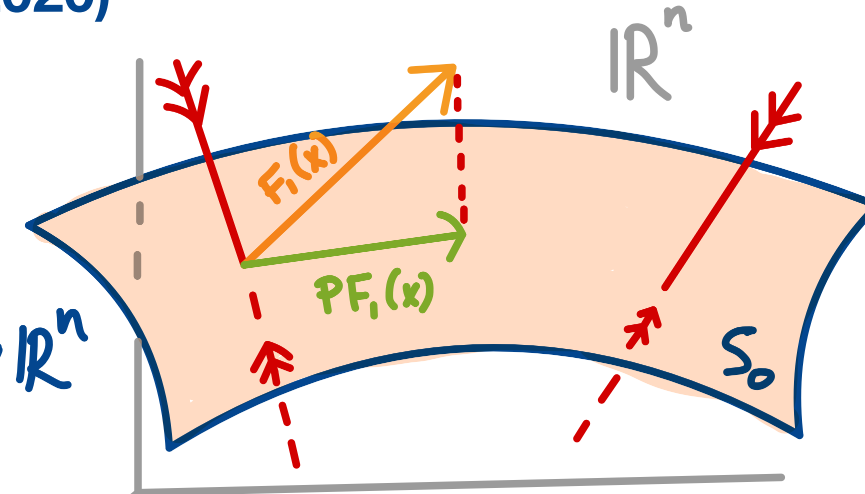


Parametrisation of the layer problem

(Feliu et al. J. Nonlinear Sci. 2020)



$$e_0 : \mathbb{R}^k \to \mathbb{R}^n$$

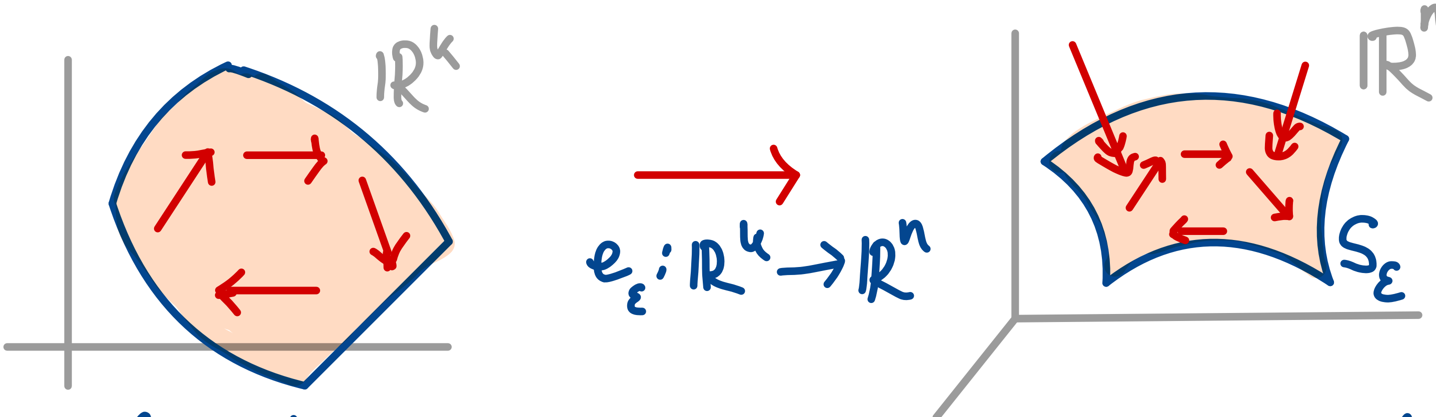


$$\frac{1}{\varepsilon}\dot{y} = f_1(y) \qquad\qquad \frac{1}{\varepsilon}\dot{x} = PF_1(x)$$

Formula:

$$De_0 \cdot f_1 = PF_1(e_0) \Leftrightarrow f_1 = De_0^{+} \cdot PF_1(e_0)$$

The (simple) idea behind the parametrisation method



$$\frac{1}{\varepsilon}\dot{y} = \frac{1}{\varepsilon} f_\varepsilon(y,\varepsilon) = f_1(y) + \varepsilon \ldots \qquad \dot{x} = F_\varepsilon(x) = F_0(x) + \varepsilon \ldots$$

Search $e_\varepsilon : \mathbb{R}^k \to \mathbb{R}^n$ **and** $f_\varepsilon : \mathbb{R}^k \to \mathbb{R}^k$ **satisfying**

$$De_\varepsilon(y) \cdot f_\varepsilon(y) = F_\varepsilon(e_\varepsilon(y)).$$

Solution: **solve iteratively by series expansion Ansatz**

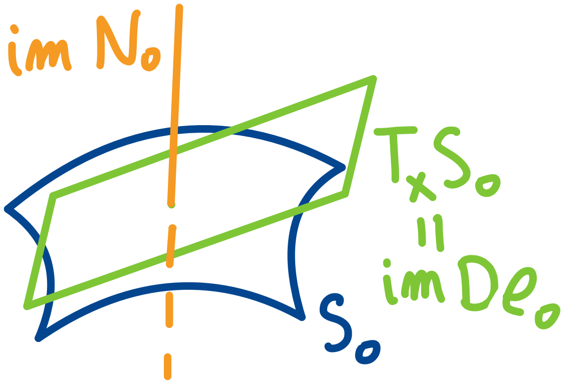
$$\begin{cases} e_\varepsilon(y) = e_0(y) + \varepsilon e_1(y) + \varepsilon^2 \cdots \\ f_\varepsilon(y) = \varepsilon f_1(y) + \varepsilon^2 f_2(y) + \varepsilon^3 \cdots \end{cases}$$

This produces:

$$\begin{cases} De_0 \cdot f_1 - DF_0(e_0)\cdot e_1 = F_1(e_0) & =: G_1 \\ De_0 \cdot f_2 - DF_0(e_0)\cdot e_2 = \cdots & =: G_2 \\ \qquad \vdots \qquad\qquad\qquad \vdots & \\ De_0 \cdot f_i - DF_0(e_0)\cdot e_i = \cdots & =: G_i \end{cases}$$

Solving the infinitesimal conjugacy equation

$$De_0 \cdot f_i - DF_0(e_0)\cdot e_i = G_i$$



Parametrise normal bundle by

$$N_0 : \mathbb{R}^k \longrightarrow L(\mathbb{R}^{n-k}, \mathbb{R}^n)$$

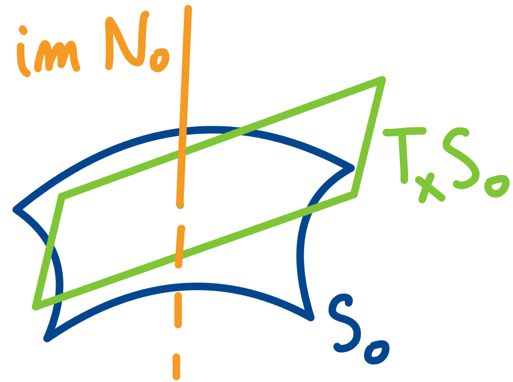
so $DF_0(e_0)\cdot N_0 = N_0 \cdot n_0$

invertible (n-k)x(n-k) matrix

Ansatz $e_i = De_0 \cdot X_i + N_0 \cdot Y_i$ **yields**

$$\underbrace{De_0 \cdot f_i}_{\in T_x S_0} - \underbrace{N_0 \cdot n_0 \cdot Y_i}_{\in \,\mathrm{im}\, N_0} = G_i$$

$$\underbrace{De_0 \cdot f_i}_{\in T_x S_0} - \underbrace{N_0 \cdot n_0 \cdot Y_i}_{\in \mathrm{im}\, N_0} = G_i$$



Projections give

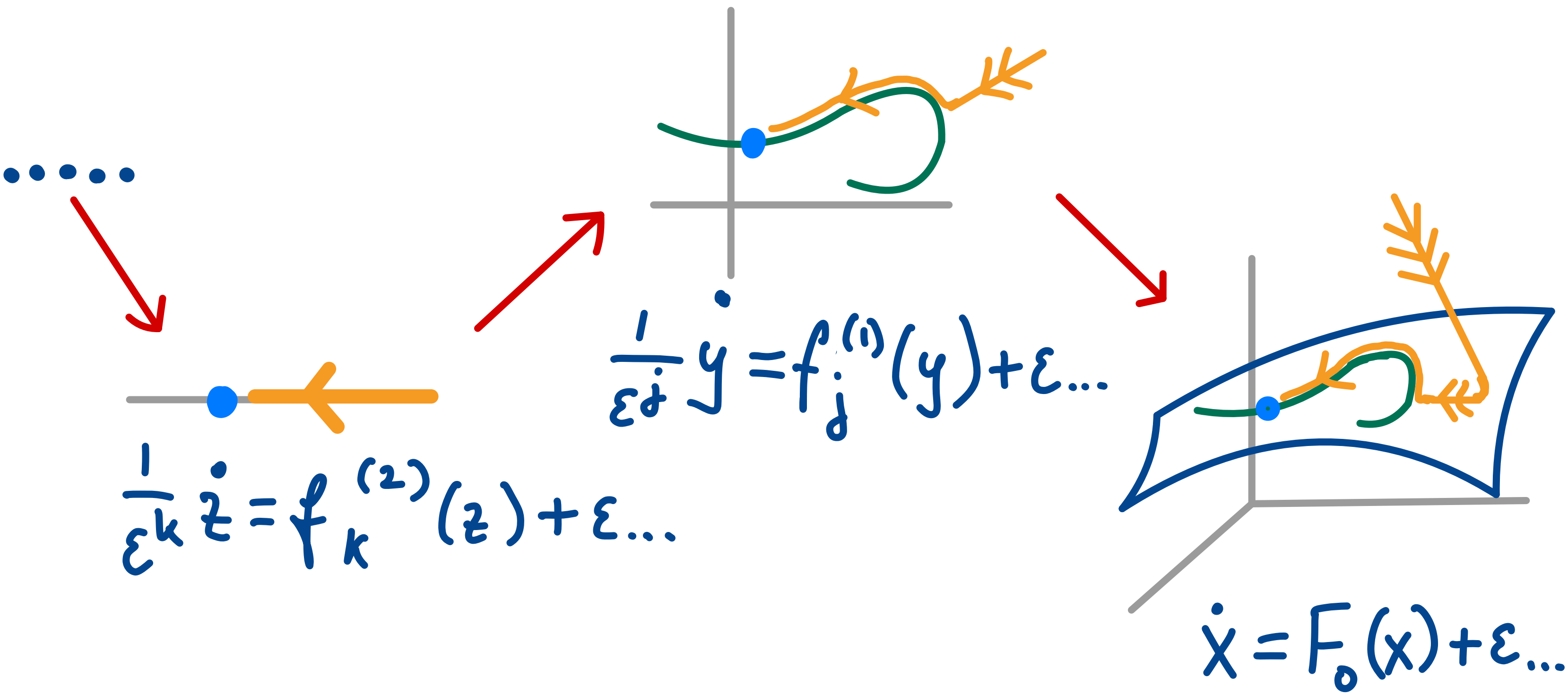
$$De_0 \cdot f_i = PG_i \quad \textbf{and} \quad N_0 \cdot n_0 \cdot Y_i = (P-1)G_i$$

For *i=1* this becomes Tikhonov's formula:

$$De_0 \cdot f_1 = PF_1$$

Compare with CSP!

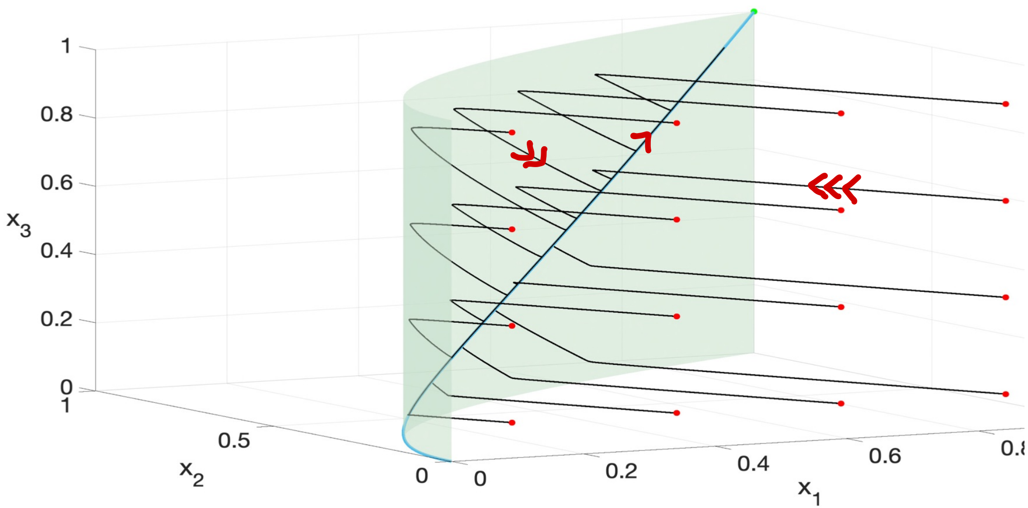
Finding hidden timescales



Application: Valorani et al. J. Comp. Phys 2005

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}(x_1 - x_2^2) + \varepsilon \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}(x_1 x_2 - x_3) + \delta \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}(x_2 x_3 - x_1)$$

$\varepsilon = 0.2$ $\delta = 0.02$



$$X_1 \underset{}{\overset{1}{\rightleftarrows}} 2X_2$$

$$X_1 + X_2 \overset{\varepsilon}{\rightleftarrows} X_3$$

$$X_2 + X_3 \overset{\delta}{\rightleftarrows} X_1$$

Analytical proof for $\delta = c\,\varepsilon^2$.

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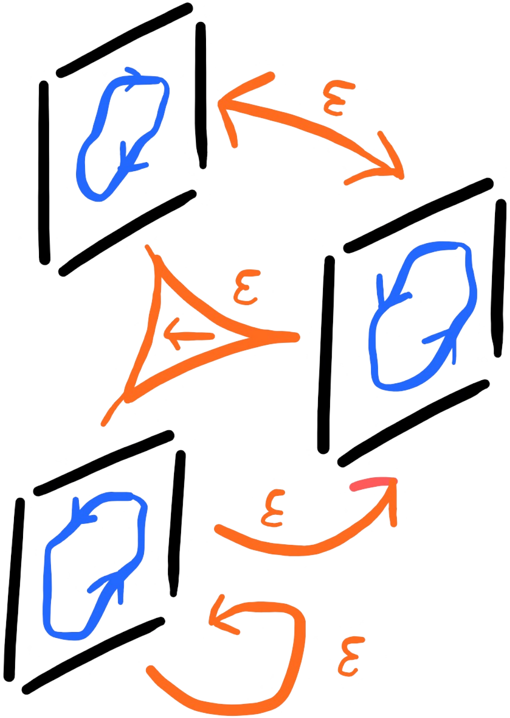
High order phase reduction



$$\dot{x}_i = \underbrace{F_i(x_i)}_{\text{has limit cycle}} + \varepsilon G_i(x_1, \ldots, x_m)$$

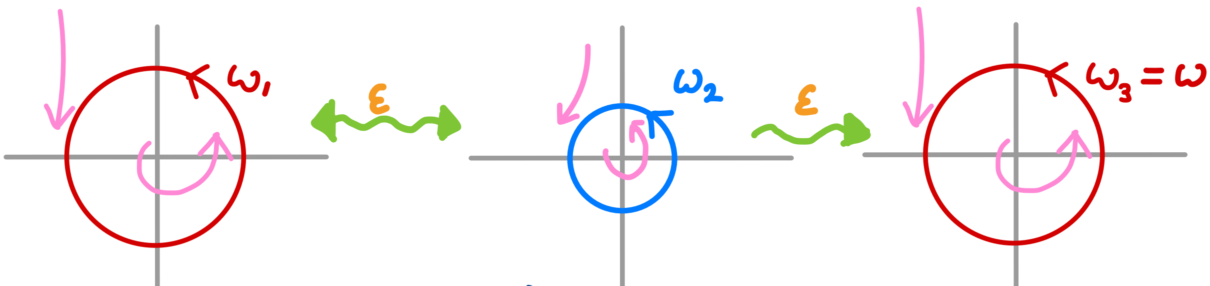
For $\varepsilon = 0$, the product of periodic orbits defines an invariant quasi-periodic torus.

By normal hyperbolicity it persists for $0 < \varepsilon \ll 1$.

Phase reduction computes/approximates the dynamics on the persisting torus.

Remote synchronisation of coupled Stuart-Landau oscillators

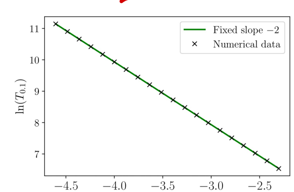
$$\begin{aligned}
\dot z_1 &= (\alpha + i\beta) z_1 + (\gamma + i\delta)|z_1|^2 z_1 + \varepsilon(z_2 - z_1)\\
\dot z_2 &= (a + ib) z_2 + (c + id)|z_2|^2 z_2 + \varepsilon(z_1 - z_2)\\
\dot z_3 &= (\alpha + i\beta) z_3 + (\gamma + i\delta)|z_3|^2 z_3 + \varepsilon(z_2 - z_3)
\end{aligned}$$



After phase reduction:

$$\begin{cases}
\dot\phi_1 = \omega_1 + \varepsilon C_1 + \varepsilon^2\{B + C\}\\
\dot\phi_2 = \omega_2 + \varepsilon C_2 + \varepsilon^2 D\\
\dot\phi_3 = \omega_1 + \underbrace{\varepsilon C_1 + \varepsilon^2\{A\sin(\phi_1 - \phi_3) + B\cos(\phi_1 - \phi_3) + C}_{\text{resonant terms only}}\}
\end{cases}$$

$$\boxed{\dot\Phi = \varepsilon^2\left(-A\sin\Phi + B(1 - \cos\Phi)\right) + \mathcal{O}(\varepsilon^3)\,, \quad \Phi := \phi_1 - \phi_3}$$



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Phase reduction with delay

$$\dot{x}_i(t) = \underbrace{F_i(x_i(t))}_{\text{has hyperbolic periodic orbit}} + \varepsilon \sum_j G_{ij}(x_j(t-\tau_{ij}), x_i(t))$$

Evolution in $C([-\tau,0],\mathbb{R}^M) \ni x_t(s)$ history segment

Strategy: Find approximate embedding

$$e_\varepsilon = e_0 + \varepsilon e_1 + \ldots : \mathbb{T}^m \longrightarrow C([-\tau,0],\mathbb{R}^M)$$

Delay-coupled SL-oscillators

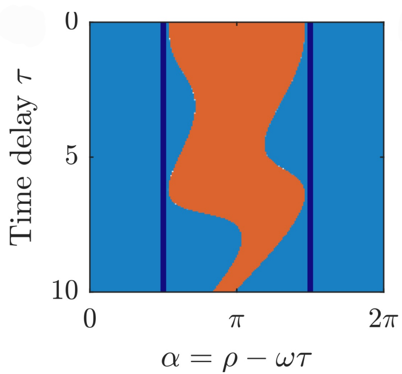
$$\dot{z}_1 = (a+ib)z_1 - |z_1|^2 z_1 + \varepsilon e^{i\rho}\left(z_2(t-\tau) - z_1\right)$$
$$\dot{z}_2 = (a+ib)z_2 - |z_2|^2 z_2 + \varepsilon e^{i\rho}\left(z_1(t-\tau) - z_2\right)$$

Phase reduction gives, for $\psi := \phi_1 - \phi_2$,

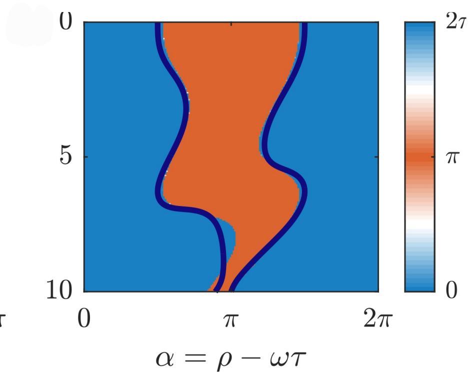
$$\dot{\psi} = \varepsilon(-2\cos\alpha)\sin\psi + \varepsilon^2(2\tau\sin\rho\sin\alpha)\sin\psi - \varepsilon^2\left(\tau + \tfrac{1}{2a}\sin^2 2\alpha\right)\sin 2\psi$$

Stability of in-phase synchrony $\{\phi_1 = \phi_2\}$.

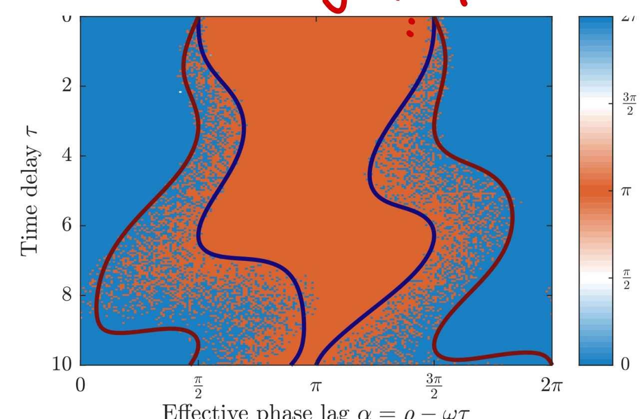
1st order reduction:



2nd order reduction:



Multistability of $\{\psi = 0\}$ & $\{\psi = \pi\}$.



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That's all Folks!